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OPTIMAL CONTROL OF ALMOST-PERIODIC MOTIONS†

A. G. Ivanov

Izhevsk

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Necessary conditions are given for an extremum (in the form of a Pontryagin maximum principle) for almost-periodic (a.p.) optimal control problems, and these conditions are automatically satisfied by convexified problems. The importance of studying the optimal control problem for a.p. motions of dynamical systems was noted in [1–3]. Such problems appear in many applications, (see for example [4–7]), and also [8] which is devoted to the problem of a.p. optimization.

1. Suppose that R^n is an n-dimensional Euclidean space, $|x|$ is the norm of the element $x \in R^n$, $\mathrm{Hom}(R^n)$ is the space of linear operators $A: R^n \to R^n$ with norm $|A| \doteq \sup_{x\neq 0}|Ax|/|x|$, and $\mathrm{comp}(R^n)$ is the collection of compact subsets of R^n. We shall denote by $S(R, Y)$ [here and below Y is any set $U \in \mathrm{comp}(R^n)$, or the space R^n or $\mathrm{Hom}(R^n)$] the collection of functions that are a.p. in the sense of Stepanov (and unless otherwise specified, we shall simply refer to "a.p. functions"). We recall [9, p. 200] that a function $f \in L_1^{\mathrm{loc}}(R, Y)$ belongs to $S(R, Y)$ if for any $\epsilon > 0$ the set

$$E_S(f, \epsilon) \doteq \{\tau \in R : \sup_{t \in R} \int_t^{t+1} |f(s+\tau) - f(s)|\, ds < \epsilon\}$$

of its almost-ϵ periods (ϵ-a.p.s) is relatively dense. To each function $f \in S(R, Y)$ there corresponds a Fourier series which can be conveniently represented in complex form

$$f(t) \sim \sum_\lambda f_\lambda e^{i\lambda t}, \quad f_\lambda \doteq M\{f(t)e^{-i\lambda t}\} \doteq \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) e^{-i\lambda t} dt$$

with the set of Fourier indices $\Lambda(f) \doteq \{\lambda \in R: |f_\lambda| > 0\}$ is not more than denumerable.

Below $\mathrm{mod}(\Delta)$ is the modulus of the set $\Delta \subseteq R$, i.e. the smallest addition group containing Δ and, if $f \in S(R, Y)$, then $\mathrm{mod}(f) \doteq \mathrm{mod}[\Lambda(f)]$ is the modulus of the function f.

The set $Q \subset S(R, Y)$ is called equally a.p. if for any $\epsilon > 0$ the set $\cap_{f \in Q} E_S(f, \epsilon)$ is relatively dense.

Suppose further that $B(R, Y)$ is a collection of a.p. functions in the sense of Bor, i.e. [9, p. 20] those functions $f \in C(R, Y)$ such that for any $\epsilon > 0$ the set

$$E_B(f, \epsilon) \doteq \{\tau \in R : \sup_{t \in R} |f(t+\tau) - f(t)| < \epsilon\}$$

is relatively dense; $B(R \times K \times U, Y)$ where $K \times U \in \mathrm{comp}(R^n \times R^n)$ is the collection of functions $f \in C(R \times K \times U, Y)$ which are a.p. with respect to t in the sense of Bor uniformly with respect to $(x, u) \in K \times U$. This means [10, p. 17] that for any $\epsilon > 0$ the set

$$\bigcap_{(x,u)\in K \times U} E_B(f(\cdot, x, u), \epsilon)$$

is relatively dense.

Suppose further that V is an open set in R^n, and that the function f: $R \times V \times U \to R^n$, which is differentiable with respect to x, satisfies the following conditions: (1) $f \in C(R \times V \times U, R^n)$,

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$f'_x \in C[R\times V\times U, \mathrm{Hom}(R^n)]$, and (2) $f\in B(R\times K\times U, R^n), f'_x\in B[R\times K\times U, \mathrm{Hom}(R^n)]$ for every compact $K\subset V$. We shall also assume that the function f_0: $R\times V\times U\to R$ also satisfies conditions 1 and 2 (with the corresponding change in dimensions).

Suppose that $\Delta\subseteq R$, then $D_1(\Delta)$ is the set of functions $u(\cdot)\in S(R, U)$ such that $\mathrm{Mod}(u)\subseteq \mathrm{Mod}(\Delta)$, and if $\psi\in C(R, R^n)\doteq C(R^n)$, then $\overline{\mathrm{orb}(\psi)}$ is the closure (in R^n) of the set $\mathrm{orb}(\psi)\doteq\{\psi(t), t\in R\}$.

Definition 1. The problem

$$J_0(x(\cdot), u(\cdot))\doteq M\{f_0(t, x(t), u(t))\}\to\inf \tag{1.1}$$

where $x(\cdot)$ is a Bor a.p. solution of the system

$$x^{\cdot}=f(t,x,u(t)),\quad u(\cdot)\in D_1(\Delta) \tag{1.2}$$

and $\overline{\mathrm{orb}(x)}\subset V$, is called an a.p. optimal control problem, and $D_1(\Delta)$ is the set of admissible (ordinary) controls (as regards the well-posedness of the problem see Sec. 2, Corollary 2.1).

We will give some necessary extremality conditions for the optimal control of a convexified (relative to the original) a.p. optimal control problem. On the importance of the enlargement (convexification) procedure applied to optimal control problems see, for example, [11–13], and in the theory of games [14, 15]. With this aim, in the following section we shall introduce the space of APM measure-valued a.p. mappings, while to conclude this section we recall the concept of exponential dichotomy.

Suppose $F\in L_1^{\mathrm{loc}}[R, \mathrm{Hom}(R^n)]$ is also integrally bounded, i.e.

$$\sup_{t\in R}\int_t^{t+1}|F(s)|\,ds<\infty$$

The system

$$x^{\cdot}=F(t)x,\quad x\in R^n \tag{1.3}$$

is exponentially dichotomous [16, 17] if there exist mutually complementary projections P_1, $P_2\in\mathrm{Hom}(R^n)$ and constants $\gamma_1, \gamma_2, \sigma_1, \sigma_2>0$ such that

$$\begin{aligned}|\Phi(t)P_1\Phi^{-1}(s)|&\leqslant\gamma_1\exp(-\sigma_1(t-s)),\quad -\infty<s\leqslant t<\infty\\ |\Phi(t)P_2\Phi^{-1}(s)|&\leqslant\gamma_2\exp(-\sigma_2(s-t)),\quad -\infty<t\leqslant s<\infty\end{aligned} \tag{1.4}$$

where $\Phi(\cdot)$ is the fundamental matrix of system (1.3). In this case the function $(t,s)\mapsto G(t,s)\in\mathrm{Hom}(R^n)$, $t, s\in R$ defined by the equalities

$$G(t,s)\doteq\chi_{(-\infty,t)}(s)\Phi(t)P_1\Phi^{-1}(s)-\chi_{(t,\infty)}(s)\Phi(t)P_2\Phi^{-1}(s)$$

where $\chi_Q(\cdot)$ is the characteristic function of the set $Q\subset R$, is called the (main) Greens function of system (1.3).

If $F\in S[R, \mathrm{Hom}(R^n)]$ and system (1.3) is exponentially dichotomous, then [17, 18] for any function $b\in S(R, R^n)$ the system $x^{\cdot}=F(t)x+b(t)$ has a unique solution $x(\cdot)$ bounded on the entire numerical axis, computed from the formula

$$x(t)=\int_R G(t,s)\,b(s)\,ds,\quad t\in R$$

and $x(\cdot)\in B(R, R^n)$.

2. Suppose $\mathrm{frm}(U)$ is the linear space of Radon measures on R^n whose basis is contained in U, and $\mathrm{rpm}(U)$ is the subset of $\mathrm{frm}(U)$ consisting of probabilistic Radon measures. We will denote by $N\doteq N[R, \mathrm{frm}(u)]$ the collection of (Lebesgue) measurable mappings μ: $R\to\mathrm{frm}(U)$ such that $\|\mu\|\doteq\mathrm{ess\,sup}_{t\in R}|\mu(t)|(U)<\infty$. [Here $|\mu(t)|(U)$ is the variation of the measure $\mu(t)\in\mathrm{frm}(U)$ and $N_1\doteq N(R, \mathrm{rpm}(U))$.] Suppose further that $|\mathrm{B}\doteq|\mathrm{B}(R\times U, R^n)$ [$|\mathrm{B}_1\doteq|\mathrm{B}(R\times U, R)$] is the collection of functions φ: $R\times U\to R^n$ such that the map $t\mapsto\varphi(t,u)$, $u\in U$ is measurable, $\varphi(t,\cdot)\in C(U, R^n)$ for almost all $t\in R$ and there exists a function $\psi_\varphi(\cdot)\in L_1(r, R)$ such that for almost all $t \in R$ the inequality $\max_{u \in U} |\varphi(t,u)| \leqslant \psi_\varphi(t)$. It is easy to show that $|\mathrm{B}$ is a linear space and

$$\|\varphi\|_{\mathrm{IB}} \doteq \int_R \max_{u \in U} |\varphi(t,u)|\, dt$$

is the norm of $\varphi \in |\mathrm{B}$. Furthermore, with minor changes to the proof of the Danford–Pettis theorem [11, p. 299] one can show that $N \cong |\mathrm{B}_1^*$ and the map $\|\cdot\|_W$: $N \to R$ defined for $\mu \in N$ by the equality

$$\|\mu\|_w \doteq \sum_{j=1}^{\infty} \frac{2^{-j}}{1 + \|\varphi_j\|_{\mathrm{IB}_1}} \left| \int_R \langle \mu(t), \varphi_j(t,u) \rangle\, dt \right|$$

$$\langle \mu(t), \varphi_j(t,u) \rangle \doteq \int_U \varphi_j(t,u)\, \mu(t)(du)$$

where $\{\varphi_1, \varphi_2, \ldots\}$ is the denumerable and everywhere dense set of functions in $|\mathrm{B}$ and is a (weak) norm in N. The space $(N, \|\cdot\|_W)$ is separable, the set $N_1 \subset (N, \|\cdot\|_W)$ is a convex compact set and if μ_j, $\mu \in N_1$, $j = 1, 2, \ldots$, then $\lim_{j\to\infty} \|\mu_j - \mu\|_W = 0$ if and only if

$$\lim_{j \to \infty} \int_R \langle \mu(t) - \mu_j(t), \varphi(t,u) \rangle\, dt = 0$$

for any function $\varphi \in |\mathrm{B}$.

Definition 2.1. [19, p. 5]. The map $\mu \in N$ is called a.p. if for any function $g \in C(U, R^n)$ the map $t \mapsto \langle \mu(t), g(u) \rangle$ belongs to $S(R, R^n)$.

The collection of all a.p. maps $\mu \in N$ ($\mu_1 \in N_1$) is denoted by APM (APM_1). We denote by $\mathrm{APM}_1^{(1)}$ the collection of those $\mu \in \mathrm{APM}_1$ such that $\mu(t) = \delta_{u(t)}$ for almost all $t \in R$ and some measurable function $u(\cdot)$: $U \to R$, where $\delta_{u(t)}$ is the Dirac measure concentrated at the point $u(t) \in U$. One can show that $\mathrm{APM}_1^{(1)} \cong S(R, U)$ and, consequently, each $u(\cdot) \in S(R, U)$ can be considered as an element of the space APM_1, identifying it with $\delta_{u(\cdot)}$.

If $\mu \in \mathrm{APM}$, then by definition the map $t \mapsto \langle \mu(t), g(u) \rangle$ belongs to $S(R, R^n)$ for every function $g \in C(U, R^n)$. Hence for every $\lambda \in R$ there exist means $A_\mu[g, \lambda] \doteq M\{\langle \mu(t), g(u) \rangle \cos \lambda t\}$, $B_\mu[g, \lambda] \doteq M\{\langle \mu(t), g(u) \rangle \sin \lambda t\}$ with

$$\langle \mu(t), g(u) \rangle \sim A_\mu[g, 0] + 2 \sum_{\lambda \neq 0} A_\mu[g, \lambda] \cos \lambda t + B_\mu[g, \lambda] \sin \lambda t \tag{2.1}$$

As has already been noted, the set $\Lambda(\mu, g) \doteq \{\lambda \in R\colon |A_\mu[g, \lambda]| + |B_\mu[g, \lambda]| > 0\}$ of Fourier indices for a.p. maps $t \mapsto \langle \mu(t), g(u) \rangle$ is not greater than denumerable, and in (2.1) it is to be understood that $A_\mu[g, \lambda] = B_\mu[g, \lambda] = 0$ if $\lambda \bar{\in} \Lambda(\mu, g)$. In [19] it was shown that for every $\lambda \in R$ one can find measures α_λ, $\beta_\lambda \in \mathrm{frm}(U)$ such that $A_\mu[g, \lambda] = \langle \alpha_\lambda, g(u) \rangle$, $B_\mu[g, \lambda] = \langle \beta_\lambda, g(u) \rangle$ for all $g \in C(U, R^n)$. Suppose now that $\{g_1, g_2, \ldots\}$ is a denumerable set that is everywhere dense in $C(U, R^n)$ and consists of continuous functions and $\Lambda(\mu) \doteq \{\lambda \in R\colon |\alpha_\lambda|(U) + |\beta_\lambda|(U) > 0\}$. It turns out [19, p. 7] that $\Lambda(\mu) = \cup_{j=1}^{\infty} \Lambda(\mu, g_j)$ (hence $\Lambda(\mu)$ is no larger than a denumerable set). $\Lambda(\mu)$ is called the set of Fourier indices of the map $\mu \in \mathrm{APM}$, and the sign-valued series

$$\alpha_0 + 2 \sum_{\lambda \neq 0} \alpha_\lambda \cos \lambda t + \beta_\lambda \sin \lambda t -$$

is its Fourier series. Here $\mathrm{Mod}(\mu) \doteq \mathrm{Mod}[\Lambda(\mu)]$ is the modulus of the map $\mu \in \mathrm{APM}$.

Theorem 2.1.† If $\varphi \in B(R \times U, R)$, then for any $\mu \in \mathrm{APM}$ the map $t \mapsto \langle \mu(t), \varphi(t,u) \rangle$ belongs to $S(R, R)$ and its modulus is contained in $\mathrm{Mod}[\Lambda(\mu) \cup \Lambda(\varphi)]$.

†IVANOV A. G., Sign-valued almost-periodic functions. Unpublished paper, Izhevsk, 1991. Deposited in VINITI 24.04.91, No. 1721–V91.

Corollary 2.1. If the function $g: R\times V\times U\to R^n$ satisfies conditions (1) and (2), and $\mu\in \mathrm{APM}$, then for every $x\in V$ the map $t\mapsto\langle\mu(t), g(t, x, u)\rangle$ belongs to $S(R, R^n)$ and, if the function $x(\cdot)\in B(R, V)$ is such that $\overline{\mathrm{orb}(x)}\subset V$, then the maps $t\mapsto\langle\mu(t), g(t, x(t), u)\rangle$, $t\mapsto\langle\mu(t), g'_x(t, x(t), u)\rangle$ belong to the spaces $S(R, R^n)$ and $S[R, \mathrm{Hom}(R^n)]$ respectively.

Suppose further that

$$D_2(\Delta)\doteq\{\mu\in \mathrm{APM}_1 : \mathrm{Mod}(\mu)\subsetneq \mathrm{Mod}(\Delta)\} \tag{2.2}$$

Definition 2.2. The problem

$$J(x(\cdot),\mu(\cdot))\doteq M\{\langle\mu(t), f_0(t,x(t),u)\rangle\}\to\inf \tag{2.3}$$

where $x(\cdot)$ is Bor a.p. solution of the a.p. system

$$\dot{x}=\langle\mu(t), f(t,x,u)\rangle\doteq\int_U f(t,x,u)\,\mu(t)(du),\quad \mu(\cdot)\in D_2(\Delta) \tag{2.4}$$

where $\overline{\mathrm{orb}(x)}\subset V$, is called a convexified problem of optimal control of a.p. motions in which any such pair $[x(\cdot), \mu(\cdot)]$ is called an admissible controlled process, and $D_2(\Delta)$ is the set of admissible controls; $H(t, x, \nu, p)\doteq -p\langle\nu, f(t, x, u)\rangle+\langle\nu, f_0(t, x, u)\rangle$, $p\in R^{n*}$, and $\nu\in \mathrm{rpm}(U)$ is the Pontryagin function for problem (2.3), (2.4).

Theorem 2.2. Suppose that $[x^\circ(\cdot), \mu^\circ(\cdot)]$ is a solution of problem (2.3), (2.4), and that the a.p. system of equations

$$\dot{y}=\langle\mu^\circ(t), f'_x(t,x^\circ(t),u)\rangle y,\quad y\in R^n \tag{2.5}$$

is exponentially dichotomous. Then for a $p(\cdot)\in B(R, R^{n*})$ which is a solution of the system of equations

$$\dot{p}=-p\langle\mu^\circ(t), f'_x(t,x^\circ(t),u)\rangle+\langle\mu^\circ(t), f'_{0x}(t,x^\circ(t),u)\rangle,\quad p\in R^{n*} \tag{2.6}$$

the Pontryagin maximum principle

$$\sup_{\nu\in D_2(\Delta)} M\{H(t,x^\circ(t),\nu(t),p(t))\}=M\{H(t,x^\circ(t),\mu^\circ(t),p(t))\} \tag{2.7}$$

is satisfied.

One can prove the following theorem (see Theorem 4.2 and Remark 1.1 in [20]).

Theorem 2.3. Equality (2.7) is satisfied if and only if for almost all $t\in R$

$$\max_{\nu\in \mathrm{rpm}(U)} H(t,x^\circ(t),\nu,p(t))=H(t,x^\circ(t),\mu^\circ(t),p(t))$$

To prove Theorem 2.2 we need the concept of a needle-shaped variation of $\mu^\circ\in D_2(\Delta)$, and also the theorem given below. But first we recall [19] that the set $Q\subset \mathrm{APM}$ is called equally a.p. if for any function $g\in C(U, R^n)$ the set $\{\langle\mu(\cdot), g(u)\rangle, \mu(\cdot)\in Q\}\subset S(R, R^n)$ is equally a.p.

Theorem 2.4.† Suppose β is a limit of an open set A of the linear normed space $(L, \|\cdot\|_L)$ and we are given a map $\alpha\to\mu(\cdot, \alpha)$ from A into APM_1. Suppose further that the function f: $R\times V\times U\to R^n$ satisfies conditions 1 and 2, that the pair $[x(\cdot, \beta), \mu(\cdot, \beta)]$ [where $\mu(\cdot, \beta)\in \mathrm{APM}_1$) is such that $x(\cdot, \beta)$ is a Bor a.p. solution of system (2.4) with $\mu(t)=\mu(t, \beta)$ and $\overline{\mathrm{orb}[x(\cdot, \beta)]}\subset V$ and, furthermore, that the a.p. system

$$\dot{y}=\langle\mu(t,\beta), f'_x(t,x(t,\beta),u)\rangle y,\quad y\in R^n$$

is exponentially dichotomous. Then, if the set $\{\mu(\cdot,\alpha), \alpha\in A\}$ is equally a.p. and $\|\mu(\cdot,$

†IVANOV A. G., Continuous dependence of an almost-periodic solution of a system of differential equations on a functional parameter—I. Unpublished paper. Deposited in VINITI 30.08.91, No. 3610–V91.

$\alpha)-\mu(\cdot,\beta)\|_w\to 0$ as $\|\alpha-\beta\|_L\to 0$, then one can find a $\gamma>0$ such that for all $\alpha\in A$ for which $\|\alpha-\beta\|_L<\gamma$ the system $x^{\bullet}=\langle\mu(t,\alpha),\ f(t,\ x,\ u)\rangle$ has a Bor a.p. solution $x(\cdot,\alpha)$ such that $\overline{\text{orb}[x(\cdot,\alpha)]}\subset V$ and, moreover, $\|x(\cdot,\alpha)-x(\cdot,\beta)\|_{C(R^n)}\to 0$ as $\|\alpha-\beta\|_L\to 0$.

3. Choose an $a>0$ such that $2\pi/a\in\text{Mod}(\Delta)$ and suppose that $\vartheta\in[0,a)$, $\alpha\in A\doteq(0,\ a-\vartheta)$ and $\nu\in D_2(\Delta)$ [see (2.2)]. Throughout the following $[x^\circ(\cdot),\ \mu^\circ(\cdot)]$ is a solution of problem (2.3), (2.4).

Definition 3.1. The map $t\mapsto\mu(t,\ \alpha)$, $\alpha\in A$, defined by

$$\mu(t,\alpha)=\begin{cases}\mu^\circ(t), & t\in\bigcup\limits_{m\in Z}[ma,(m+1)a]\backslash T_{m,\alpha,\vartheta}\\ \nu(t), & t\in\bigcup\limits_{m\in Z}T_{m,\alpha,\vartheta}\doteq\bigcup\limits_{m\in Z}[ma+\vartheta,ma+\vartheta+\alpha)\end{cases}\tag{3.1}$$

is called a needle-like variation of $\mu^\circ(\cdot)\in D_2(\Delta)$.

The following theorem follows directly from the definition.

Theorem 3.1. The family of maps $\{\mu(\cdot,\ \alpha),\ \alpha\in A\}$ defined by formulae (3.1) belongs to $D_2(\Delta)$, and is equally a.p. and $\|\mu(\cdot,\ \alpha)-\mu^\circ(\cdot)\|_w\to 0$ for $\alpha\downarrow 0$.

Corollary 3.1. If system (2.5) is exponentially dichotomous, then for all sufficiently small $\alpha\in A$ system (2.4) has for $\mu(t)=\mu(t,\ \alpha)$ a Bor a.p. solution $x(\cdot,\ \alpha)$ such that $\overline{\text{orb}[x(\cdot,\alpha)]}\subset V$ and

$$\lim_{\alpha\downarrow 0}\|x^\circ(\cdot)-x(\cdot,\alpha)\|_{C(R^n)}=0\tag{3.2}$$

Throughout the following, assuming $F(t)=\langle\mu^\circ(t),f'_x(t,x^\circ(t),u)\rangle$ for system (2.5), we will retain the notation used in the definition of exponential dichotomy of system (1.3). Furthermore, the set $K\in\text{comp}(R^n)$ is such that $\overline{\text{orb}(x^\circ)}\subset K\subset V$. By Corollary 3.1 we can find an interval $A_1\subset A$ such that for all $\alpha\in A_1$, $\overline{\text{orb}[\Delta x(\cdot,\alpha)]}\subset K$, where $\Delta x(\cdot,\ \alpha\doteq x^\circ(\cdot)-x(\cdot,\ \alpha)$.

Lemma 3.1. There exists an interval $A_0\subset A$ such that the set $\{\alpha^{-1}\Delta x(\cdot,\ \alpha),\ \alpha\in A_0\}\subset B(R,\ R^n)$ is uniformly bounded.

Proof. In the paper cited in the first footnote it is shown that one can find an interval $A_2\subseteq A$ such that for all $\alpha\in A_2$ the function $\Delta x(\cdot,\ \alpha)\in B(R,\ K)$ satisfies the equation

$$z=\int_R G(t,s)[h_\alpha(s,z)+g(s,z)]\,ds\tag{3.3}$$

$$h_\alpha(t,z)\doteq\langle\mu^\circ(t)-\mu(t,\alpha),f(t,x^\circ(t)-z,u)\rangle$$

$$g(t,z)\doteq\langle\mu^\circ(t),f(t,x^\circ(t),u)-f(t,x^\circ(t)-z,u)\rangle-\langle\mu^\circ(t),f'_x(t,x^\circ(t),u)\rangle z$$

Because $f'_x\in B[R\times K\times U,\ \text{Hom}(R^n)]$, then (see [10]) one can find an $\alpha_0>0$ such that

$$|f'_x(t,x_1,u)-f'_x(t,x_2,u)|<\frac{3}{4}\left(\frac{\gamma_1}{\sigma_1}+\frac{\gamma_2}{\sigma_2}\right)^{-1}\tag{3.4}$$

for all $(t,\ x_j,\ u)\in R\times K\times U$, $j=1,\ 2$, if $|x_1-x_2|<\alpha_0$. Because for $\alpha\in A_2$ [see (3.3)]

$$\frac{\Delta x(t,\alpha)}{\alpha}=\frac{1}{\alpha}\int_R G(t,s)\,h_\alpha(s,\Delta x(s,\alpha))\,ds+$$

$$+\int_R G(t,s)\left\langle\mu^\circ(s),\int_0^1(f'_x(s,x^\circ(s)-\theta\Delta x(s,\alpha),u)-f'_x(s,x^\circ(s),u))\,d\theta\right\rangle\frac{\Delta x(s,\alpha)}{\alpha}\,ds\tag{3.5}$$

then from (3.1), the definitions of the functions $h_\alpha(t, z)$ and (3.4) for all $\alpha \in A_0 \doteq (0, \alpha_0) \cap A_2$ we obtain the proof of

$$\sup_{\alpha \in A_0} \left\| \frac{\Delta x(\cdot, \alpha)}{\alpha} \right\|_{C(R^n)} < \Gamma_0 \left(\frac{\gamma_1}{1 - \exp(-\sigma_1 a)} + \frac{\gamma_2}{1 - \exp(-\sigma_2 a)} \right) \tag{3.6}$$

$$\Gamma_0 \doteq \sup \{ |f(t, x, u)|, (t, x, u) \in R \times K \times U \} \tag{3.7}$$

Lemma 3.2. The following limit equality holds as $\lim_{\alpha \downarrow 0}$, $\alpha \in A_0$

$$\lim_{\alpha \downarrow 0} \left\| \frac{\Delta x(\cdot, \alpha)}{\alpha} - \frac{1}{\alpha} \int_R G(\cdot, s) \langle \mu^\circ(s) - \mu(s, \alpha), f(s, x^\circ(s), u) \rangle ds \right\|_{C(R^n)} = 0 \tag{3.8}$$

The proof of Lemma 3.2 follows directly from the inclusions $f \in B(R \times K \times U, R^n)$, $f'_x \in B[R \times K \times U, \mathrm{Hom}(R^n)]$, equality (3.5) and inequality (3.6), and so we omit it.

Throughout the following (see 1.4)

$$\begin{aligned}
&Z_+ \doteq \{0, 1, \ldots\} \\
&X_\tau(t, s) \doteq \Phi(t + \tau)\Phi^{-1}(s + \tau), \quad P_j(t, s) \doteq \Phi(t) P_j \Phi^{-1}(s), \quad j = 1, 2 \\
&\Delta f(t, \nu) \doteq \langle \nu(t) - \mu^\circ(t), f(t, x^\circ(t), u) \rangle, \quad \eta(t) \doteq \langle \mu(t), f'_{0x}(t, x^\circ(t), u) \rangle \\
&\Gamma_1 \doteq \sup \{ |f'_{0x}(t, x, u)|, (t, x, u) \in R \times K \times U \}, \quad \Gamma \doteq \max(\Gamma_0, \Gamma_1)
\end{aligned} \tag{3.9}$$

We note that $\eta(\cdot) \in S(R, R)$ and because $\|\mu^\circ\| = 1$, $\Gamma \geq \Gamma_1 \geq \operatorname{ess\,sup}_{t \in R} |\eta(t)|$. Furthermore, suppose that for $s \in [0, a]$

$$\xi^{(1)}_{m,k}(s) \doteq \int_{ma}^{(m+1)a} \eta(t) P_1(t, t + s - ka - a) \Delta f(t + s - ka - a, \nu) dt \tag{3.10}$$

$$\xi^{(2)}_{m,k}(s) \doteq \int_{ma}^{(m+1)a} \eta(t) P_2(t, t + s + ka) \Delta f(t + s + ka, \nu) dt \tag{3.11}$$

Lemma 3.3. For every $k \in Z_+$ the sets $\{\xi^{(j)}_{m,k}, m \in Z_+\}$, $j = 1, 2$ are contained in $C([0, a], R^n)$ and are equicontinuous.

Proof. Suppose $\omega(h)$ $(h > 0)$ is equal to

$$\sup \{ |X_r(0, s_1) - X_r(0, s_2)|, (r, s_j) \in R \times [ka, (k+1)a], \quad j = 1, 2, |s_1 - s_2| \leq h \}$$

Because $\Delta f(\cdot, \nu) \in S(R, R^n)$, then [9, p. 201]

$$\lim_{h \downarrow 0} \left(\sup_{r \in R} \frac{1}{a} \int_r^{r+a} |\Delta f(t + h, \nu) - \Delta f(t, \nu)| dt = 0 \tag{3.12}$$

Now, using the fact that $\omega(h) \to 0$ as $h \downarrow 0$ and the easily obtained inequality [here see (3.7) and (3.9)]

$$|\xi^{(2)}_{m,k}(s + h) - \xi^{(2)}_{m,k}(s)| \leq a \gamma_2 \Gamma \sup_{r \in R} \frac{1}{a} \int_r^{r+a} |\Delta f(t + h, \nu) - \Delta f(t, \nu)| dt + 2\gamma_2 a \Gamma^2 \omega(h)$$

we find that the set $\{\xi^{(2)}_{m,k}, m \in Z_+\} \subset C([0, a], R^n)$ is also equicontinuous. We can similarly prove the corresponding assertion for the set $\{\xi^{(1)}_{m,k}, m \in Z_+\}$.

Suppose further that $\Xi^{(1)}_{k,m,\alpha}(\vartheta)$ and $\Xi^{(2)}_{k,m,\alpha}(\vartheta)$ are respectively equal to

$$\begin{aligned}
&\int_{ma}^{(m+1)a} \eta(t) \Big[\frac{1}{\alpha} \int_{T_k^-} P_1(t, t + s) \Delta f(t + s, \nu) ds - \\
&- P_1(t, t - ka - a + \vartheta) \Delta f(t - ka - a + \vartheta, \nu) \Big] dt
\end{aligned}$$

$$\int_{ma}^{(m+1)a} \eta(t) \Big[\frac{1}{\alpha} \int_{T_k^+} P_2(t, t + s) \Delta f(t + s, \nu) ds - P_2(t, t + ka + \vartheta) \Delta f(t + ka + \vartheta, \nu) \Big] dt$$

where [see (3.1)]

$$T_k^- \doteq T_{-(k+1),\alpha,\vartheta}, \quad T_k^+ \doteq T_{k,\alpha,\vartheta} \tag{3.13}$$

Lemma 3.4. For every $k\in Z_+$ and $\vartheta\in[0,a]$

$$\lim_{\substack{\alpha\downarrow 0\\ \alpha\in A_0}} \left(\sup_{m\in Z_+} |\Xi_{k,m,\alpha}^{(j)}(\vartheta)|\right)=0, \quad j=1,2 \tag{3.14}$$

Proof. Changing the order of integration in each $\Xi_{k,m,\alpha}^{(j)}(\vartheta)$, and using (3.10) and (3.11), we obtain the inequality

$$|\Xi_{k,m,\alpha}^{(j)}(\vartheta)| \leqslant \frac{1}{\alpha}\int_{\vartheta}^{\vartheta+\alpha} |\xi_{m,k}^{(j)}(s)-\xi_{m,k}^{(j)}(\vartheta)|\,ds, \quad j=1,2$$

Then (3.14) follows from Lemma 3.3.

Throughout the following

$$L(t,\vartheta)\doteq\sum_{k=0}^{\infty}(P_1(t,t-ka-a+\vartheta)\Delta f(t-ka-a+\vartheta,\bar{\nu})-$$
$$-P_2(t,t+ka+\vartheta)\Delta f(t+ka+\vartheta,\nu))$$

Using the "a.p. along the diagonal" property (see the paper mentioned in the second footnote) the maps $(t,s)\mapsto P_j(t,s)$, $j=1,2$ and the inclusion $\Delta f(\cdot,\nu)\in S(R,R^n)$ one can show that the family of maps $\{L(\cdot,\vartheta),\ \vartheta\in[0,a]\}$ belongs to $S(R,R^n)$ and is uniformly bounded and equally a.p. Furthermore, we have the following theorem.

Theorem 3.2. For almost all $\vartheta\in[0,a]$, $a\in A_0$

$$\lim_{\alpha\downarrow 0}\left(\sup_{m\in Z_+}\left|\int_{ma}^{(m+1)a}\eta(t)\left(\frac{\Delta x(t,\alpha)}{\alpha}-L(t,\vartheta)\right)dt\right|\right)=0 \tag{3.15}$$

Proof. Suppose $\gamma\doteq\max(\gamma_1,\gamma_2)$, $\sigma\doteq\max(\sigma_1,\sigma_2)$. Because

$$\Xi_\alpha(\vartheta)\doteq\sum_{k=0}^{\infty}\left(\sup_{m\in Z_+}|\Xi_{k,m,\alpha}^{(1)}(\vartheta)|+\sup_{m\in Z_+}|\Xi_{k,m,\alpha}^{(2)}(\vartheta)|\right)\leqslant 8\gamma a\Gamma^2\sum_{k=0}^{\infty}e^{-\sigma ka}$$

then for a given $\epsilon>0$ one can find a $k_0\geqslant 1$ such that for all $(\alpha,\vartheta)\in A_0\times[0,a)$

$$\Xi_\alpha(\vartheta)<\frac{\epsilon}{2}+\sum_{k=0}^{k_0}\left(\sup_{m\in Z_+}|\Xi_{k,m,\alpha}^{(1)}(\vartheta)|+\sup_{m\in Z_+}|\Xi_{k,m,\alpha}^{(2)}(\vartheta)|\right) \tag{3.16}$$

Then, because for $\alpha\in A_0$

$$\sup_{m\in Z_+}\left|\int_{ma}^{(m+1)a}\eta(t)\left(\frac{\Delta x(t,\alpha)}{\alpha}-L(t,\vartheta)\right)dt\right|\leqslant\Xi_\alpha(\vartheta)+$$
$$+a\Gamma\left\|\frac{\Delta x(\cdot,\alpha)}{\alpha}-\int_R G(\cdot,s)\langle\mu^\circ(s)-\mu(s,\alpha),f(s,x^\circ(s),u)\rangle ds\right\|_{C(R^n)}$$

(3.15) follows from (3.16) and (3.14), (3.18).

We put $\Delta f_0(t,\nu)\doteq\langle\nu(t)-\mu^\circ(t),f_0(t,x^\circ(t),u)\rangle$. By Corollary 2.1 $\Delta f(\cdot,\nu)\in S(R,R)$. Hence (see the paper mentioned in the first footnote) for almost all $\vartheta\in[0,a]$ there exists

$$\lim_{q\to\infty}\frac{1}{qa}\sum\Delta f_0(\vartheta+ma,\nu)\doteq\xi(\vartheta)$$

and the map $\vartheta\mapsto\xi(\vartheta)$ is measurable and bounded. Above and below the sums are taken from $m=0$ to $m=q-1$.

Lemma 3.5. We have the equality

$$\lim_{\alpha\downarrow 0}\int_0^a \left| \lim_{q\to\infty}\frac{1}{qa}\Sigma\left(\frac{1}{\alpha}\int_{\vartheta}^{\vartheta+\alpha}\Delta f_0(t+ma,\nu)\,dt-\Delta f_0(\vartheta+ma,\nu)\right)\right| d\vartheta=0$$

Proof. Because the function $\Delta f_0(\cdot,\nu)\in S(R,R)$, it obeys a limiting equality similar to (3.12). Hence

$$\int_0^a \left| \lim_{q\to\infty}\frac{1}{qa}\Sigma\left(\frac{1}{\alpha}\int_{\vartheta}^{\vartheta+\alpha}\Delta f_0(t+ma,\nu)\,dt-\Delta f_0(\vartheta+ma,\nu)\right)\right| d\vartheta\leqslant$$

$$\leqslant \lim_{q\to\infty}\frac{1}{qa}\Sigma\int_0^a\left(\frac{1}{\alpha}\int_0^{\alpha}|\Delta f_0(t+\vartheta+ma,\nu)-\Delta f_0(\vartheta+ma,\nu)|\,dt\right)d\vartheta\leqslant$$

$$\leqslant \sup_{0\leqslant t\leqslant\alpha}\left(\sup_{r\in R}\frac{1}{a}\int_r^{r+a}|\Delta f_0(s+t,\nu)-\Delta f_0(s,\nu)|\,ds\to 0\right.$$

when $\alpha\downarrow 0$, as was required.

Corollary 3.1. One can find a sequence $\{\alpha_j\}_{j=1}^{\infty}$ such that $\alpha_j\downarrow 0$ as $j\to\infty$ and for almost all $\vartheta\in[0,a]$, for example for $\vartheta\in\Xi$, we have the equality

$$\lim_{j\to\infty}\left(\lim_{q\to\infty}\frac{1}{qa}\Sigma\left(\frac{1}{\alpha_j}\int_{\vartheta}^{\vartheta+\alpha_j}\Delta f_0(t+ma,\nu)\,dt-\Delta f_0(\vartheta+ma,\nu)\right)\right)=0 \tag{3.17}$$

4. We will give a proof of Theorem 2.2. Take any $\vartheta\in\Xi\cap[0,a)$ (see Corollary 3.1). By the definition of A_0, for all $\alpha\in A_0$ the pair $[x(\cdot,\alpha),\mu(\cdot,\alpha)]$ is an admissible controlled process of problem (2.3), (2.4). Because $[x^\circ(\cdot),\mu^\circ(\cdot)]$ is a solution of this problem, then, putting $\omega\doteq\alpha+\vartheta$ and using (3.1) and (3.13), we have

$$0\leqslant\alpha^{-1}(J(x(\cdot,\alpha),\mu(\cdot,\alpha))-J(x^\circ(\cdot),\mu^\circ(\cdot))=$$

$$=\frac{1}{\alpha}\lim_{q\to\infty}\frac{1}{qa}\Sigma\int_{ma}^{(m+1)a}(\langle\mu(t,\alpha),f_0(t,x(t,\alpha),u)\rangle-\langle\mu^\circ(t),f_0(t,x^\circ(t),u)\rangle)\,dt=$$

$$=\lim_{q\to\infty}\frac{1}{qa}\Sigma\left\{\frac{1}{\alpha}\int_{ma}^{ma+\vartheta}\langle\mu^\circ(t),f_0(t,x(t,\alpha),u)-f_0(t,x^\circ(t),u)\rangle\,dt+\right.$$

$$+\frac{1}{\alpha}\int_{T_m^+}(\langle\nu(t),f_0(t,x(t,\alpha),u)\rangle-\langle\mu^\circ(t),f_0(t,x^\circ(t),u)\rangle)\,dt+$$

$$\left.+\frac{1}{\alpha}\int_{ma+\omega}^{(m+1)a}\langle\mu^\circ(t),f_0(t,x(t,\alpha),u)-f_0(t,x^\circ(t),u)\rangle\,dt\right\}=$$

$$=\lim_{q\to\infty}\frac{1}{qa}\Sigma\left\{\int_{ma}^{(m+1)a}\langle\mu^\circ(t),\int_0^1 f'_{0x}(t,x^\circ(t)+\theta\Delta x(t,\alpha),u)\,d\theta\rangle\frac{\Delta x(t,\alpha)}{\alpha}\,dt+\right.$$

$$+\frac{1}{\alpha}\int_{T_m^+}\langle\nu(t),f_0(t,x(t,\alpha),u)-f_0(t,x^\circ(t),u)\rangle\,dt+\frac{1}{\alpha}\int_{T_m^+}\Delta f_0(t,\nu)\,dt+$$

$$\left.+\int_{ma+\omega}^{(m+1)a}\langle\mu^\circ(t),\int_0^1 f'_{0x}(t,x^\circ(t)+\theta\Delta x(t,\alpha),u)\,d\theta\rangle\frac{\Delta x(t,\alpha)}{\alpha}\,dt\right\}$$

We now put $\alpha=\alpha_j$, where $\{\alpha_j\}_{j=1}^{\infty}$ is the sequence from Corollary 3.1, by virtue of which, starting from some j_0, all $\alpha_j\in A_0$. Passing to the limit as $j\to 0$ limit in the relation obtained above, using equalities (3.17), (3.15) and (3.12), the restrictions on f_0, the inclusions μ°, $\nu\in D_2(\Delta)$ and inequality (3.6), and find that

$$0\leqslant\lim_{q\to\infty}\frac{1}{qa}\sum_{m=0}^{q-1}\left(\Delta f_0(\vartheta+ma,\nu)+\int_{ma}^{(m+1)a}\eta(t)L(t,\vartheta)\,dt\right)$$

Integrating the latter inequality over ϑ from 0 to a [see the definitions of $\eta(t)$ and $L(t,\vartheta)$], we obtain

$$M\{\langle\nu(t)-\mu^\circ(t), f_0(t, x^\circ(t), u)\rangle\} + M\{\langle\mu^\circ(t), f'_{0x}(t, x^\circ(t), u)\rangle y(t)\} \geqslant 0 \tag{4.1}$$
$$y(t) = \int_R G(t, \vartheta)\, \Delta f(\vartheta, \nu)\, d\vartheta$$

Because $p(\cdot)$ and $y(\cdot)$ are Bor a.p. solutions of system (2.6) and $y^\bullet = \langle\mu^\circ(t), f'_x[t, x^\circ(t), u]\rangle y + \Delta f(t, \nu)$ $(y \in R^n)$, respectively, we have

$$\frac{d}{dt}(p(t)y(t)) = p(t)\,\Delta f(t, \nu) + \langle\mu^\circ(t), f'_{0x}(t, x^\circ(t), u)\rangle y(t) \tag{4.2}$$

Since $\|p\|_{C(R^{n*})}$, $\|y\|_{C(R^n)} < \infty$, we have $M\{d/dt[p(t)y(t)]\} = 0$. Consequently [see (4.2)] $M\{\langle\mu^\circ(t), f'_{0x}[t, x^\circ(t), u]\rangle y(t)\} = -M\{p(t)\Delta f(t, \nu)\}$. Hence [see (4.1) and the definition of the Pontryagin function] for all $\nu \in D_2(\Delta)$, $M\{H[t, x^\circ(t), \mu^\circ(t), p(t)]\} \geqslant M\{H[t, x^\circ(t), \nu(t), p(t)]\}$, which is equivalent to (2.7). Theorem 2.2 is proved.

Remark 4.1. One can show that the exponential dichotomy condition on system (2.5) in Theorem 2.2 is important.

In conclusion we will prove the following theorem.

Theorem 4.1. Suppose $[x^\circ(\cdot), \mu^\circ(\cdot)]$ is a solution of problem (2.3), (2.4) and $\mu^\circ(\cdot) \in D_2(\Delta)\backslash D_1(\Delta)$ (recall that $S(R, U) \cong \mathrm{APM}_1^{(1)}$). Then one can find [19] a sequence $\{u_j\}_{j=1}^{\infty} \subset D_1(\Delta)$ such that $\|\mu^\circ(\cdot) - \delta_{u_j(\cdot)}\|_w \to 0$ as $j \to \infty$ and for any function $\varphi \in B(R \times U, R^n)$

$$\lim_{j\to\infty} M\{\varphi(t, u_j(t))\} = M\{\langle\mu^\circ(t), \varphi(t, u)\rangle\} \tag{4.3}$$

Furthermore, it turns out (see the paper mentioned in the second footnote) that for all sufficiently large j the system

$$x^\bullet = f(t, x, u_j(t)) \doteq \langle\delta_{u_j(t)}, f(t, x, u)\rangle$$

will have a Bor a.p. solution $x_j(\cdot)$ such that $\overline{\mathrm{orb}[x_j(\cdot)]} \subset V$ and

$$\lim_{j\to\infty} \|x^\circ(\cdot) - x_j(\cdot)\|_{C(R^n)} = 0 \tag{4.4}$$

Then from (4.3), (4.4) and the restrictions on the function f_0 we obtain $\lim_{j\to\infty} J_0[x_j(\cdot), u_j(\cdot)] = J[x^\circ(\cdot), \mu^\circ(\cdot)]$, as was required.

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